PLASTIC ZONE AROUND A ROUND HOLE IN A PLANE WITH A NONUNIFORM BASIC STRESSED STATE

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Special cases of the problem of determining the plastic zone around a round hole when the basic stressed state is a polynomial function of the coordinates were considered in [1-9], but a correct solution was not obtained. Accurate solution of this problem with a polynomial distribution of the basic stresses has been given in general form in [10], although works continue to appear [11] in which erroneous solutions are repeated. In this article by using the results from [10] a new method is suggested for finding coefficients of the transformation function and an example is given for accurate solution of the problem of determining the plastic zone around a circular hole with a specific quadratic distribution of the basic stresses.

Let in some region of an elastic body, which is under plane strain conditions or in a plane stressed state, principal stresses be prescribed by Kolosov–Muskhelishvili functions [12]

$$\Phi_0(z) = \sum_{j=0}^{m} a_j z^j, \quad \Psi_0(z) = \sum_{j=0}^{m} b_j z^j, \tag{1}$$

where coefficients are the known constants. By selecting these functions in the form of (1) it is possible to obtain the solution for problems which are of practical interest [4, 6, 13].

We make in this region a circular hole $x^2 + y^2 \leqslant R^2$ to whose contour we apply constant forces $\sigma_r = p,\ \tau_{r\vartheta} = 0,\ r = R$ (r, ϑ are polar coordinates). We assume that around the hole an axisymmetrical biharmonic stressed state is realized [10]

$$\begin{gathered}(\sigma_r + \sigma_\vartheta)/2 = p + \varepsilon k(1 - k_1) + \varepsilon k \ln(r^2 R^{-2}),\\ (\sigma_\vartheta - \sigma_r)/2 + i\tau_{r\vartheta} = \varepsilon k(1 - k_1 R^2 r^{-2}),\ \varepsilon = \pm 1,\ k > 0,\ k_1 \leqslant 1.\end{gathered} \tag{2}$$

Stresses (2) satisfy boundary conditions at the hole contour, equilibrium equations, and the condition of plasticity $[(\sigma_\vartheta - \sigma_r)/2]^2 + \tau_{r\vartheta}^2 = k^2(1 - k_1 R^2 r^{-2})^2$. With $k_1 = 0$ the well-known Tresk–St Venant plasticity condition is obtained. With special selection of k_1 stresses (2) also satisfy other plasticity conditions [10].

In the elastic zone stresses may be presented in terms of the functions

$$\Phi(z) = \Phi_0(z) + \Phi_1(z),\ \Psi(z) = \Psi_0(z) + \Psi_1(z) \tag{3}$$

($\Phi_1(z)$, $\Psi_1(z)$ characterize additional stresses caused by presence of a hole and a plastic zone). We designate the boundary between the plastic and elastic zones in terms of L. It is also necessary to find holomorphic functions $\Phi_1(z)$, $\Psi_1(z)$ so that L stresses equate to expressions (2) and in the vicinity of an infinitely distant point there is [10, 12]

$$\Phi_1(z) = O(1/z^2),\ \Psi_1(z) = O(1/z^2). \tag{4}$$

There are the equations [12]

$$\begin{gathered}(\sigma_x + \sigma_y)/2 = \Phi(z) + \overline{\Phi(z)} = (\sigma_r + \sigma_\vartheta)/2,\\ (\sigma_y - \sigma_x)/2 + i\tau_{xy} = \bar{z}\Phi'(z) + \Psi(z) = [(\sigma_\vartheta - \sigma_r)/2 + i\tau_{r\vartheta}]e^{-2i\vartheta}.\end{gathered} \tag{5}$$

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Since $e^{-2i\vartheta}=\bar{z}/z$, $r^2 = z\bar{z}$, then by substituting expressions (2) and (3) in (5) we obtain for additional functions $\Phi_1(z)$, $\Psi_1(z)$ boundary conditions in L

$$\Phi_1(z) + \overline{\Phi_1(z)} = p + \varepsilon k(1-k_1) + \varepsilon k \ln\frac{z\bar{z}}{R^2} - [\Phi_0(z) + \overline{\Phi_0(z)}], \quad z \in L,$$
$$\bar{z}\Phi_1'(z) + \Psi_1(z) = \varepsilon k\left(1 - k_1\frac{R^2}{z\bar{z}}\right)\frac{\bar{z}}{z} - [\bar{z}\Phi_0'(z) + \Psi_0(z)]. \tag{6}$$

With $|z| \to \infty$ condition (4) should be satisfied.

Let the function

$$z = Rc\omega(\zeta) = Rc\zeta\sum_{n=0}^{\infty}\frac{c_n}{\zeta^n}, \quad c = |c| > 0, \quad c_0 = 1 \tag{7}$$

reflect conformally the region $|\zeta| > 1$ in an infinite region outside unknown contour L. In (7) it is possible to assume that $c = |c| > 0$, since rotation of the coordinate system by angle arg c does not change the boundary conditions (6). We specify that

$$\Phi_2(\zeta) = k^{-1}\Phi_1[Rc\omega(\zeta)], \quad \Psi_2(\zeta) = k^{-1}\Psi_1[Rc\omega(\zeta)];$$
$$\alpha_j = k^{-1}a_j(Rc)^j = \alpha_{(j)}c^j, \quad \beta_j = k^{-1}b_j(Rc)^j = \beta_{(j)}c^j, \ j = \overline{0, m}, \tag{8}$$

and taking account of (4) and (7) with large $|\zeta|$ there should be

$$\Phi_2(\zeta) = O(1/\zeta^2), \quad \Psi_2(\zeta) = O(1/\zeta^2). \tag{9}$$

Here by using (1), (7), and (8) we rewrite boundary conditions (6) for functions $\Phi_2(\zeta)$, $\Psi_2(\zeta)$ in the form

$$\Phi_2(t) + \overline{\Phi_2(t)} = p/k + \varepsilon(1-k_1) + \varepsilon\ln c^2 + \varepsilon\ln\omega(t)\overline{\omega(t)} -$$
$$-\left[\sum_{j=0}^{m}\alpha_j\omega^j(t) + \sum_{j=0}^{m}\bar{\alpha}_j\overline{\omega^j(t)}\right], \quad |t| = 1,$$
$$\frac{\overline{\omega(t)}}{\omega'(t)}\Phi_2'(t) + \Psi_2(t) = \varepsilon\frac{\overline{\omega(t)}}{\omega(t)} - \frac{\varepsilon k_1}{c^2\omega^2(t)} - \left[\overline{\omega(t)}\sum_{j=1}^{m}j\alpha_j\omega^{j-1}(t) + \sum_{j=0}^{m}\beta_j\omega^j(t)\right]. \tag{10}$$

In (10) by values of functions in the vicinity of $|t| = 1$ we understand limiting values of functions with the approach of ζ from the region $|\zeta| > 1$ to points of the circle $|t| = 1$.

Accurate solution of boundary conditions (9) and (10) in a general form is given in [10]. In the case of m = 2 the solution of boundary probelms (9) and (10) is written as in [10]:

$$\Phi_2(\zeta) = \varepsilon\ln\frac{\omega(\zeta)}{\zeta} - \Big\{\alpha_1[\omega(\zeta) - c_1 - \zeta] + \alpha_2[\omega^2(\zeta) - (c_1^2 + 2c_2) -$$
$$-2c_1\zeta - \zeta^2] + \frac{\bar{\alpha}_1}{\zeta} + \bar{\alpha}_2\left(\frac{2\bar{c}_1}{\zeta} + \frac{1}{\zeta^2}\right)\Big\},$$
$$\Psi_2(\zeta) = \frac{-\varepsilon k_1}{c^2\omega^2(\zeta)} - \beta_0 - \beta_1\omega(\zeta) - \beta_2\omega^2(\zeta) + \frac{1}{\omega'(\zeta)}\left[\frac{\varepsilon}{\zeta^2} - \bar{\alpha}_1\left(\frac{\bar{c}_1}{\zeta^2} + \frac{1}{\zeta^3}\right) -\right.$$
$$-2\bar{\alpha}_2\left(\frac{\bar{c}_1^2 + \bar{c}_2}{\zeta^2} + \frac{2\bar{c}_1}{\zeta^3} + \frac{1}{\zeta^4}\right) + \beta_0 + \beta_1(c_1 + \zeta) +$$
$$\left. + \beta_2(c_1^2 + c_2 + 2c_1\zeta + \zeta^2)\right]; \tag{11}$$

$$\omega(\zeta) = \zeta + c_1 + \frac{c_2A_2\zeta^4 + (\varepsilon c_1 - \bar{A}_1)\zeta^3 - (\bar{A}_1c_1 + \bar{A}_2 + \bar{B}_1)\zeta^2 - (\bar{A}_2c_1 + \bar{B}_2)\zeta - \bar{B}_3}{\zeta(A_2\zeta^4 + A_1\zeta^3 - \varepsilon\zeta^2 + \bar{A}_1\zeta + \bar{A}_2)}, \tag{12}$$

where

$$A_1 = \alpha_{(1)}c + 2\alpha_{(2)}c^2c_1,\quad A_2 = 2\alpha_{(2)}c^2, \tag{13}$$
$$B_1 = \beta_0 + \beta_{(1)}cc_1 + \beta_{(2)}c^2(c_1^2 + c_2),\quad B_2 = \beta_{(1)}c + 2\beta_{(2)}c^2c_1,\ B_3 = \beta_{(2)}c^2;$$

$$p/k + \varepsilon(1 - k_1) + \varepsilon \ln c^2 - \{\alpha_0 + \bar{\alpha}_0 + (\alpha_{(1)}c_1 + \bar{\alpha}_{(1)}\bar{c}_1)c + \\ + [\alpha_{(2)}(c_1^2 + 2c_2) + \bar{\alpha}_{(2)}(\bar{c}_1^2 + 2\bar{c}_2)]c^2\} = 0. \tag{14}$$

Coefficients c_1 and c_2 in Eqs. (11)-(14) are determined by conditions which mean that the fraction in (12) should not have any singular points with $|\zeta| \geq 1$. Coefficient c is found from Eq. (14). As noted in [10], in the preceeding works [1-9, 11] before solution of the problem it is always that $c_1 = 0$, and in the equations for c in the form of (14) there are no terms which contain c outside the sign of the logarithm and condition (9) is not fulfilled. A method is suggested in [10] for obtaining equations for c_1 and c_2 consisting of requiring conformity to zero of the numerator and denominator in (12) with $|\zeta| \geq 1$. In this article we assume somewhat differently.

It is easy to prove that if ζ_q is the root of the polynomial in the denominator of (12), then $\zeta_{2+q} = 1/\bar{\zeta}_q,\ q = 1, 2,$ is also a root. If $|\zeta_q| \leq 1$, then $|\zeta_{2+q}| = 1/|\bar{\zeta}_q| \geqslant 1$, i.e., there is always one root within a single circle, and the other is outside a single circle, or the root $\zeta_q = 1/\bar{\zeta}_q$ is in a single circle.

By comparing expressions (7) and (12) we find that coefficients c_n of an expansion into a series of function $\omega(\zeta)$ in the vicinity of an infinitely distant point satisfy a recurrent set of equations

$$\begin{gathered} A_2c_3 + A_1c_2 - \varepsilon c_1 + \bar{A}_1 = 0, \\ A_2c_4 + A_1c_3 - \varepsilon c_2 + \bar{A}_1c_1 + \bar{A}_2 + \bar{B}_1 = 0, \\ A_2c_5 + A_1c_4 - \varepsilon c_3 + \bar{A}_1c_2 + \bar{A}_2c_1 + \bar{B}_2 = 0, \\ A_2c_6 + A_1c_5 - \varepsilon c_4 + \bar{A}_1c_3 + \bar{A}_2c_2 + \bar{B}_3 = 0, \\ A_2c_{n+3} + A_1c_{n+2} - \varepsilon c_{n+1} + \bar{A}_1c_n + \bar{A}_2c_{n-1} = 0,\ n = \overline{4, \infty}. \end{gathered} \tag{15}$$

In the case m = 1, i.e., with $\alpha_{(2)} = 0,\ \beta_{(2)} = 0$, we deduce that

$$\omega(\zeta) = \zeta + c_1 + \frac{(\varepsilon c_1 - \bar{\alpha}_{(1)}c)\zeta^2 - (\bar{\alpha}_{(1)}cc_1 + \bar{\beta}_0 + \bar{\beta}_{(1)}c\bar{c}_1)\zeta - \bar{\beta}_{(1)}c}{\zeta(\alpha_{(1)}c\zeta^2 - \varepsilon\zeta + \bar{\alpha}_{(1)}c)}; \tag{16}$$

$$p/k + \varepsilon(1 - k_1) + \varepsilon \ln c^2 - [\alpha_0 + \bar{\alpha}_0 + (\alpha_{(1)}c_1 + \bar{\alpha}_{(1)}\bar{c}_1)c] = 0; \tag{17}$$

$$\begin{gathered} \alpha_{(1)}cc_2 - \varepsilon c_1 + \bar{\alpha}_{(1)}c = 0, \\ \alpha_{(1)}cc_3 - \varepsilon c_2 + \bar{\alpha}_{(1)}cc_1 + \bar{\beta}_0 + \bar{\beta}_{(1)}c\bar{c}_1 = 0, \\ \alpha_{(1)}cc_4 - \varepsilon c_3 + \bar{\alpha}_{(1)}cc_2 + \bar{\beta}_{(1)}c = 0, \\ \alpha_{(1)}cc_{n+2} - \varepsilon c_{n+1} + \bar{\alpha}_{(1)}cc_n = 0,\ n = \overline{3, \infty}. \end{gathered} \tag{18}$$

First we consider Eqs. (16)-(18). Since $\omega(\zeta)$ should not have singular points with $|\zeta| \geq 1$, apart from a pole with $\zeta = \infty$, then we take the solution of system (18) in the form

$$c_0 = q_1 + \delta_0 = 1,\quad c_1 = q_1\zeta_1 + \delta_1,\quad c_2 = q_1\zeta_1^2 + \delta_2,\quad c_n = q_1\zeta_1^n,\ n = \overline{3, \infty}, \\ |\zeta_1| < 1 \tag{19}$$

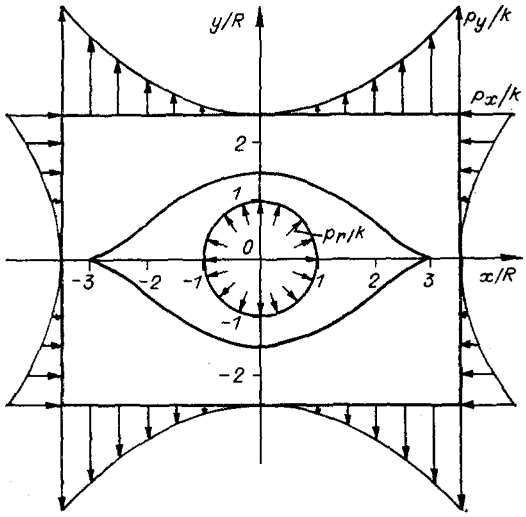

Fig. 1

(q_1, δ_0, δ_1, δ_2, ζ_1 are still undetermined values). By substituting (19) in system (18) we obtain

$$\begin{gathered}\alpha_{(1)}c\delta_2 - \varepsilon\delta_1 + \bar{\alpha}_{(1)}c\delta_0 = 0,\\ -\varepsilon\delta_2 + \bar{\alpha}_{(1)}c\delta_1 + \bar{\beta}_0 + \bar{\beta}_{(1)}c\bar{c}_1 = 0,\end{gathered} \tag{20}$$

$$\begin{gathered}\bar{\alpha}_{(1)}c\delta_2 + \bar{\beta}_{(1)}c = 0;\\ \alpha_{(1)}c\zeta_1^2 - \varepsilon\zeta_1 + \bar{\alpha}_{(1)}c = 0,\ |\zeta_1| < 1.\end{gathered} \tag{21}$$

From the first equation in (19) we have $q_1 = 1 - \delta_0$, and from Eqs. (20) successively we find that

$$\begin{gathered}\delta_2 = -\bar{\beta}_{(1)}/\bar{\alpha}_{(1)},\ \delta_1 = -(-\varepsilon\delta_2 + \bar{\beta}_0 + \bar{\beta}_{(1)}c\bar{c}_1)/(\bar{\alpha}_{(1)}c),\\ \delta_0 = -(\alpha_{(1)}c\delta_2 - \varepsilon\delta_1)/(\bar{\alpha}_{(1)}c).\end{gathered} \tag{22}$$

To the right in (22) there is c_1, but for c_1 there is the second equation of (19):

$$c_1 = (1 - \delta_0)\zeta_1 + \delta_1; \tag{23}$$

and by substituting expression (22) in (23) and considering (21) we write

$$c_1 = \left\{1 + \frac{\alpha_{(1)}}{\bar{\alpha}_{(1)}}\left[\frac{\bar{\beta}_0}{\bar{\alpha}_{(1)}c} + \frac{\bar{\beta}_{(1)}}{\bar{\alpha}_{(1)}}\left(\bar{c}_1 + \frac{\alpha_{(1)}}{\bar{\alpha}_{(1)}}\zeta_1\right)\right]\zeta_1\right\}\zeta_1. \tag{24}$$

This equation is equivalent to the corresponding equation in [10]. Thus, for unknowns c, c_1, ζ_1 we have a set of three equations (17), (21), (24).

By substituting (19) in (7) and summing the series we obtain instead of expression (16) the following:

$$\omega(\zeta) = \zeta + c_1 + \frac{\delta_2}{\zeta} + \frac{c_2 - \delta_2}{\zeta - \zeta_1}. \tag{25}$$

All of the coefficients in (25) are determined entirely by solving set (17), (21), (24) with respect to c, c_1, ζ_1, taking account of Eqs. (19) and (22). An example of solving this system and plotting the elastoplastic boundary L by Eq. (25) for specific values of prescribed external parameters is given in [10].

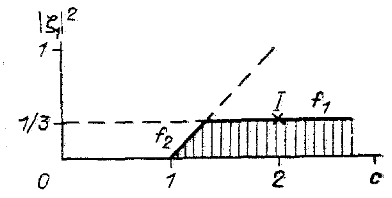

Fig. 2

Now we return to system (15) and we take its solution in the form

$$
\begin{gathered}
c_{-1}=q_1+q_2+\delta_{-1}=0,\ c_0=q_1\zeta_1+q_2\zeta_2+\delta_0=1, \\
c_1=q_1\zeta_1^2+q_2\zeta_2^2+\delta_1,\ |\zeta_1|<1,\ |\zeta_2|<1,\ \zeta_1\neq\zeta_2, \\
c_2=q_1\zeta_1^3+q_2\zeta_2^3+\delta_2,\quad c_n=q_1\zeta_1^{n+1}+q_2\zeta_2^{n+1},\ n=\overline{3,\infty}.
\end{gathered} \tag{26}
$$

In the first two equations of (26) we find that

$$
q_1=\frac{-\zeta_2\delta_{-1}-(1-\delta_0)}{\zeta_2-\zeta_1},\ q_2=\frac{1-\delta_0+\zeta_1\delta_{-1}}{\zeta_2-\zeta_1}. \tag{27}
$$

By substituting (27) in the third and fourth equations of (26) we have

$$
\begin{gathered}
c_1=\delta_{-1}\zeta_1\zeta_2+(1-\delta_0)(\zeta_1+\zeta_2)+\delta_1, \\
c_2=\delta_{-1}\zeta_1\zeta_2(\zeta_1+\zeta_2)+(1-\delta_0)[(\zeta_1+\zeta_2)^2-\zeta_1\zeta_2]+\delta_2.
\end{gathered} \tag{28}
$$

Equations for $\delta_{-1}, \delta_0, \delta_1, \delta_2, \zeta_1, \zeta_2$ are obtained by substituting (26) in (15):

$$
\begin{gathered}
A_1\delta_2-\varepsilon\delta_1+\bar{A}_1\delta_0+\bar{A}_2\delta_{-1}=0, \\
-\varepsilon\delta_2+\bar{A}_1\delta_1+\bar{A}_2\delta_0+\bar{B}_1=0,
\end{gathered} \tag{29}
$$

$$
\begin{gathered}
\bar{A}_1\delta_2+\bar{A}_2\delta_1+\bar{B}_2=0,\ \bar{A}_2\delta_2+\bar{B}_3=0; \\
A_2\zeta_{1,2}^4+A_1\zeta_{1,2}^3-\varepsilon\zeta_{1,2}^2+\bar{A}_1\zeta_{1,2}+\bar{A}_2=0.
\end{gathered} \tag{30}
$$

Successively from (29) we find that

$$
\begin{gathered}
\delta_2=-\bar{B}_3/\bar{A}_2, \\
\delta_1=-(\bar{A}_1\delta_2+\bar{B}_2)/\bar{A}_2, \\
\delta_0=-(-\varepsilon\delta_2+\bar{A}_1\delta_1+\bar{B}_1)/\bar{A}_2,\ \delta_{-1}=-(A_1\delta_2-\varepsilon\delta_1+\bar{A}_1\delta_0)/\bar{A}_2.
\end{gathered} \tag{31}
$$

In view of (13) in the right-hand part of (31) there are coefficients c_1 and c_2, then by substituting (31) in (28) we obtain equations for them. Thus, values of $c, c_1, c_2, \zeta_1, \zeta_2$ required by us are found from set of Eqs. (14), (28), (30). If in (31) we substitute expression (28) then we shall have a set of equations for $\delta_{-1}, \delta_0, \delta_1$, and here it is also necessary to take account of (14) and (30).

By summing series (7) with coefficients (26) we write instead of expression (12)

$$
\omega(\zeta)=\zeta+c_1+\frac{\delta_2}{\zeta}+\frac{q_1\zeta_1^3}{\zeta-\zeta_1}+\frac{q_2\zeta_2^3}{\zeta-\zeta_2}, \tag{32}
$$

where all of the coefficients are determined by solving system (14), (28), (30) and taking account of Eqs. (27) and (31).

If it is assumed that prescribed parameters are such that polynomial (30) may have a multiple root $\zeta_1 = \zeta_2$, $|\zeta_1| < 1$, then instead of (26) the solution of (15) should be taken in the form

$$c_{-1} = q_1 + \delta_{-1} = 0,\ c_0 = (q_1 + q_2)\zeta_1 + \delta_0 = 1,$$
$$c_1 = (q_1 + 2q_2)\zeta_1^2 + \delta_1,\ \ c_2 = (q_1 + 3q_2)\zeta_1^3 + \delta_2,\ c_n = [q_1 + (n+1)q_2]\zeta_1^{n+1},\qquad (33)$$
$$n = \overline{3, \infty}.$$

From the first two equations of (33) we have

$$q_1 = -\delta_{-1},\ q_2 = (1 - \delta_0)/\zeta_1 + \delta_{-1}, \qquad (34)$$

and equations for c_1 and c_2 here take the form

$$c_1 = \delta_{-1}\zeta_1^2 + 2(1 - \delta_0)\zeta_1 + \delta_1,\ \ c_2 = 2\delta_{-1}\zeta_1^3 + 3(1 - \delta_0)\zeta_1^2 + \delta_2, \qquad (35)$$

which also follows from (28) with $\zeta_1 = \zeta_2$.

With substitution of (33) in system (15) we obtain for $\delta_{-1}, \delta_0, \delta_1, \delta_2$ Eqs. (29) or (31), and instead of (30)

$$A_2\zeta_1^4 + A_1\zeta_1^3 - \varepsilon\zeta_1^2 + \overline{A}_1\zeta_1 + \overline{A}_2 = 0; \qquad (36)$$

$$4A_2\zeta_1^3 + 3A_1\zeta_1^2 - 2\varepsilon\zeta_1 + \overline{A}_1 = 0,\ \ |\zeta_1| < 1, \qquad (37)$$

where condition (37) provides multiplicity of root ζ_1 for polynomial (36). This is possible if coefficients in (36) are connected by additional relationships. From the Viéte equation for (36) we have

$$\zeta_1 + 1/\overline{\zeta}_1 = -A_1/2A_2,\ \zeta_1/\overline{\zeta}_1 = \overline{A}_1/A_1; \qquad (38)$$

$$(-A_1/2A_2)^2 + 2\overline{A}_1/A_1 = -\varepsilon/A_2,\ (\overline{A}_1/A_1)^2 = \overline{A}_2/A_2. \qquad (39)$$

With fulfilment of (39) from (38) it is possible to find ζ_1. Equations (36) and (37) are equivalent to relationships (38) and (39).

By summing series (7) with coefficients (33) we obtain instead of (12)

$$\omega(\zeta) = \zeta + c_1 + \frac{\delta_2}{\zeta} + \frac{c_2 - \delta_2}{\zeta - \zeta_1} + \frac{q_2\zeta_1^4}{(\zeta - \zeta_1)^2},$$

here all of the coefficients are entirely determined by solving system (14), (35), (36), (37) and taking account of Eqs. (34) and (31).

We consider an example. Let the following parameters be prescribed

$$a_0 = 0,\ b_0 = 0,\ a_1 = 0,\ b_1 = 0,\ b_2 = 0,\ a_2 = \overline{a}_2, \qquad (40)$$

then the main stresses will be these:

$$\sigma_x = -4a_2y^2,\ \sigma_y = 4a_2x^2,\ \tau_{xy} = 0.$$

From (8), (13), (40) we have

$$\alpha_0 = 0,\ \beta_0 = 0,\ \alpha_{(1)} = 0,\ \beta_{(1)} = 0,\ \alpha_{(2)} = k^{-1}a_2R^2 = \overline{\alpha}_{(2)},\ \ \beta_{(2)} = 0,$$
$$A_2 = 2\alpha_{(2)}c^2 = \overline{A}_2,\ A_1 = 2\alpha_{(2)}c^2c_1 = A_2c_1,\ B_1 = 0,\ B_2 = 0,\ B_3 = 0. \qquad (41)$$

Equations (14) and (30) take the form

$$p/k + \varepsilon(1-k_1) + \varepsilon \ln c^2 - \alpha_{(2)}\left(c_1^2 + 2c_2 + \bar{c}_1^2 + 2\bar{c}_2\right)c^2 = 0; \tag{42}$$

$$\zeta_{1,2}^4 + c_1\zeta_{1,2}^3 - \frac{\varepsilon}{2\alpha_{(2)}c^2}\zeta_{1,2}^2 + \bar{c}_1\zeta_{1,2} + 1 = 0. \tag{43}$$

Taking account of (41) from (31), (28), (27), and (32)

$$\begin{gathered}\delta_2 = 0,\ \delta_1 = 0,\ \delta_0 = 0,\ \delta_{-1} = 0;\\ c_1 = \zeta_1 + \zeta_2,\ c_2 = (\zeta_1 + \zeta_2)^2 - \zeta_1\zeta_2;\\ q_1 = -(\zeta_2 - \zeta_1)^{-1},\ q_2 = (\zeta_2 - \zeta_1)^{-1};\end{gathered} \tag{44}$$

$$\omega(\zeta) = \zeta + \zeta_1 + \zeta_2 + \frac{1}{\zeta_1 - \zeta_2}\left(\frac{\zeta_1^3}{\zeta - \zeta_1} - \frac{\zeta_2^3}{\zeta - \zeta_2}\right). \tag{45}$$

By excluding ζ_1, and ζ_2 from (43) and (44) we write

$$\bar{c}_1 + c_1\left(-c_1^2 + 3c_2 - \varepsilon/A_2\right) = 0,\quad 1 - \left(c_1^2 - c_2\right)\left(c_1^2 + c_2 - \varepsilon/A_2\right) = 0. \tag{46}$$

It is possible to solve the system for three unknowns c, c_1, c_2: (42), (46).

From the Viéte equation for Eq. (43) taking account of (44) we have the relationships

$$\begin{gathered}2c_1 + \bar{c}_1/(\zeta_1\zeta_2) = 0,\ \zeta_1\zeta_2 + (c_1\bar{c}_1 + 1)/(\zeta_1\zeta_2) = -\varepsilon/(2\alpha_{(2)}c^2),\\ \zeta_1\zeta_2 = \overline{\zeta_1\zeta_2}.\end{gathered} \tag{47}$$

From the first equation of (47) two versions follow: $c_1 = \zeta_1 + \zeta_2 = 0$ and $c_1 \neq 0,\ \zeta_1\zeta_2 = \pm 1/2$ The second version does not suit us since here as the proof shows it is impossible to satisfy univalent conditions for function $\omega(\zeta)$. There remains the other version $c_1 = 0$, $\xi_2 = -\zeta_1$. Then from the second equation of (47) we find that

$$\zeta_1^2 = \frac{\varepsilon}{4\alpha_{(2)}c^2}\left[1 - \sqrt{1 - (4\alpha_{(2)}c^2)^2}\right], \tag{48}$$

and there should be $2\sqrt{|\alpha_{(2)}|}c < 1$. Equation (45) takes the form

$$\omega(\zeta) = \zeta + \frac{\zeta_1^2}{2}\left(\frac{1}{\zeta - \zeta_1} + \frac{1}{\zeta + \zeta_1}\right) = \frac{\zeta^3}{\zeta^2 - \zeta_1^2}. \tag{49}$$

Function $\omega(\zeta)$ of (49) is univalent with $|\zeta| > 1$, and boundary L entirely embraces a circular hole if the following equalities are fulfilled

$$|\zeta_1|^2 \leqslant 1/3,\ 1 + |\zeta_1|^2 \leqslant c. \tag{50}$$

From the first inequality of (50) taking account of (48) we have $2\sqrt{|\alpha_{(2)}|}c \leqslant \sqrt{3/5}$.

Equation (42) taking account of (44) and (48) is written as:

$$\varepsilon p/k - k_1 = -\left[\ln c^2 + \sqrt{1 - (4\alpha_{(2)}c^2)^2}\right]. \tag{51}$$

Relationship (51) is an equation for c. If c is specified, then this will be an equation for p/k (what pressure should be applied to the hole contour so that the zone is plastic and limitation (50) is fulfilled).

Let $|\zeta_1|^2 = 1/3,\ c = 2$, here $4|\alpha_{(2)}| = 0.15$, $\varepsilon p/k - k_1 = -2.1863$, $\sigma_x/k = -0.15(y/R)^2$, $\sigma_y/k = 0.15\,(x/R)^2$, $\tau_{xy} = 0$. From (49) we obtain an equation for contour L in parametric form

$$\frac{x}{R}=\frac{3\cos\varphi-2\cos^3\varphi}{1/3+\sin^2\varphi},\ \frac{y}{R}=\frac{2\sin^3\varphi}{1/3+\sin^2\varphi},\quad \varphi\in[0,2\pi]. \tag{52}$$

The form of contour (52) is presented in Fig. 1 where $p_i=\sigma_{ij}n_j$ (n_j are components of the normal to the area). The region of parameters (50) with which a solution of (49) exists is shown in Fig. 2 (hatched). Contour L in Fig. 1 is plotted from parameters corresponding to point I in Fig. 2. If parameters relate to straight line f_2, then boundary L will touch the contour of the hole, and if it relates to straight line f_1, then L will be a point of return as in Fig. 1.

LITERATURE CITED

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10. N. I. Ostrosablin, Plane Elastoplastic Distribution of Stresses around Circular Holes [in Russian], Nauka, Novosibirsk (1984).
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13. S. P. Timoshenko and J. Goodyear, Elasticity Theory [Russian translation], Nauka, Moscow (1979).

REVIEWER'S COMMENTS

I have taken the liberty of adding some references which are directly related to the subject of this article:

14. B. D. Annin and G. P. Cherepanov, The Elastoplastic Problem [in Russian], Nauka, Novosibirsk (1983).
15. D. D. Ivlev and L. V. Ershov, Method of Disturbances in Elastoplastic Body Theory [in Russian], Nauka, Moscow (1978).
16. E. M. Morozov and G. P. Nikishkov, The Finite Element Method in Fracture Mechanics [in Russian], Nauka, Moscow (1980).
17. S. N. Atluri (ed.), Numerical methods in Fracture Mechanics [Russian translation], Mir, Moscow (1989).

The errors included in previous works [1-9, 11] are entirely correctly criticized by Ostrasablin. However, the criticism is insufficient. In studying specific elastoplastic problems it is necessary to be aware of the following.

1. In order to solve the elastoplastic problem after satisfying differential equations and boundary conditions it is necessary to check fulfilment of inequalities which emerge from the condition of plasticity in the "elastic" region since analytical methods do not guarantee fulfillment of this inequality in the "elastic" region of the solution. It is possible to argue that in the problem of a plastic zone close to a hole with a polynomial distribution of stresses at a distance from the hole the plasticity condition is always infringed in the "elastic" region for so-called accurate solutions if the power of the polynomial m is greater or equal to one. This was well known by Galin who did not attach particular

importance to these "accurate" solutions, and he considered them an illustration of the method itself (for a circular hole with plane strain). All of the "accurate" solutions [1-11] with $m \geq 1$ (and of the present article) do not satisfy the plasticity condition.

2. All accurate analytical solutions for elastoplastic problems obtained in [1-11] and in this article are obtained easily by the method of functional equations [14].

3. Unfortunately, accurate determination of the boundary of the plastic region is not always important since plastic strain in a considerable part of the plastic zone may be negligibly small. In this case an approximate solution is much more useful which "embraces" a "deeper" zone in which marked plastic strains occur.

4. The search for accurate analytical solutions of elastoplastic problems, which is not a very promising scientific subject to which extensive literature has been devoted (see e.g. [6, 9-11, 14, 15], although all of the available accurate solutions may be counted on the fingers of one hand. Computer methods are more promising, the most effective of which are given in [16, 17].

February 25, 1989 G. P. Cherepanov

AUTHOR'S REPLY

1. It is well known that an infinite plane with a hole is idealization of the problem for a finite region with a hole in the case when the hole dimensions are small compared with those of the region. Idealization is performed in order not to solve the more complex problem for a doubly connected region. The solution for an infinite plane with a hole will be approximate for a finite region with a hole, although the solution itself for an idealized boundary problem may be found accurately. Therefore a solution for an infinite plane with a hole may be used only in a certain finite region around the hole. This is noted for example in [10, 12], and in this article it is not specially mentioned since this is a generally accepted approach. If in a finite region without a hole the main stressed state is polynomial, then evidently coefficients a_j and b_j in (1) should be such that a condition of plasticity is not achieved in the region in question. For an example from the article we have the principal stresses: $(\sigma_y - \sigma_x)/2k + i\tau_{xy}/k = 2\alpha_{(2)}r^2/R^2$, and the Tresk plasticity condition will not be achieved if $2|\alpha_{(2)}|r^2/R^2 < 1$. This inequality determines the ratio between the value of coefficient $|\alpha_{(2)}|$ and the relative size of the hole and a finite plate in which the solution should be considered. For a numerical example we have $r^2 < R^2/2|\alpha_{(2)}| = R^2/2 \cdot 0.0375 = 13.33R^2$, i.e., in this region it is permissible to use a solution for an infinite plate with a hole. Additional stresses caused by presence of a hole and a plastic zone around it are given complex potentials $\Phi_1(z), \Psi_1(z)$ and with large r they are of the order $O(1/r^2)$, i.e., they decrease rapidly and they not strongly alter the region in which it is possible to use the solution for an infinite plate with a hole.

2. Although the method of functional equations from [14] makes it possible sometimes to obtain a solution by selecting $\omega(\zeta)$ in the form of a polynomial or a rational function and selection of unknown coefficients, the equation for transformation of function $\omega(\zeta)$ obtained in [10, 14] solves the problem for the general case and in this article it does not complicate the solution compared with the method of functional equations.

3. Here the statically determined problem is solved when stresses and the elastoplastic boundary are independent of equations for strains. In order to determine strains or displacements in the plastic zone it is necessary to solve its set of equations and to have for it boundary conditions at the elastoplastic boundary (see e.g. [10, 15]). For this it is also necessary to know the elastoplastic boundary at which values of strains or displacements from the direction of the elastic region will be unknown. Before solving equations for strains we do not know where these strains are in the plastic zone. If we have accurate expressions for the elastoplastic boundary and strains in it, then this is no worse than any approximate expressions.

4. A specific problem is solved in this article and the future for these investigations is not discussed. The author agrees with the fact that numerical methods are used extensively and they make it possible to solve more complex problems for which it is impossible to find an analytical solution. However, accurate analytical solutions are also important and they may for example be used as a test for checking numerical methods.

Works [14, 15] may be included in the literature cited since they are devoted to studying elastoplastic problems, but all the same the problem of the present article is not considered directly in them and therefore they were not cited.

Numerical methods are used in [16, 17] for solving mechanics problems. Numerical methods are not used or discussed in this article. Inclusion of them in the literature cited is considered to be superfluous.

April 7, 1989 N. I. Ostrosablin